Lecture 15

Wednesday, February 19, 2020 5:35 AM

Pseudoconvexity.

Recall. If $\delta(z) = \max(|z_1|, \dots, |z_n|)$ and $\delta(z, \mathbb{C}^n \setminus \Omega) = \inf_{w \in \mathbb{C}^n \setminus \Omega} \delta(z - w)$, then:

Prop 1 If $\Omega \subseteq \mathbb{C}^n$ is d.o.holom., $K \subset\subset \Omega$, $f \in \mathcal{O}(\Omega)$ s.t. $|f(z)| \le \delta(z, \mathbb{C}^n \setminus \Omega)$ on K, then $|f(z)| \le \delta(z, \mathbb{C}^n \setminus \Omega)$ in $\hat{K}_\Omega$.

Rem. The conclusion can be proved w/ any continuous function $\delta(z)$ s.t. $\delta(tz) = |t|\delta(z)$, $t \in \mathbb{C}$. E.g. $\delta(z) = \left(\sum_{j=1}^n |z_j|^p\right)^{1/p}$, L^p-norm.

Thm 5. Let $\Omega \subseteq \mathbb{C}^n$ d.o.holom. and $\delta(z) > 0$ as in Rem above. Then, the fcn $u(z) = -\log \delta(z, \mathbb{C}^n \setminus \Omega)$ is continuous and $PSH(\Omega)$.

Pf. Cont. of u follows from continuity of $\delta(z, \mathbb{C}^n \setminus \Omega)$. The latter is Ex.. Pick $z^0 \in \Omega$, $w \in \mathbb{C}^n$, $\tau \in \mathbb{C}$, $|\tau| \le \varepsilon$, and consider $v(\tau) = u(z^0 + \tau w)$. Let $K = \{z^0 + \tau w \in \Omega : |\tau| = \varepsilon\} \subset\subset \Omega$. Clearly, by Max. Mod. Princ. in τ-plane, we conclude $\{z^0 + \tau w \in \Omega : |\tau| \le \varepsilon\} \subseteq \hat{K}_\Omega$. Let $p(\tau)$ be any holom. polynomial in $\tau \in \mathbb{C}$, and $Q(z)$ a holom. poly. in $z \in \mathbb{C}^n$ s.t. $p(\tau) = Q(z^0 + \tau w)$. If $v(\tau) \le \operatorname{Re} p(\tau)$ on $|\tau| = \varepsilon$, then $e^{v(\tau)} \le |e^{p(\tau)}|$ on $|\tau| = \varepsilon$ or, equivalently,

$$\delta(z^0 + \tau w, \mathbb{C}^n \setminus \Omega)^{-1} \le |e^{Q(z^0 + \tau w)}|, \quad |\tau| = \varepsilon \qquad (1)$$

If we set $f(z) = e^{-Q(z)} \in \mathcal{O}(\Omega)$, then inverting (1) $\Rightarrow$

$$|f(z)| \le \delta(z, \mathbb{C}^n \setminus \Omega), \quad z \in K.$$

By Prop 1, $|f(z)| \le \delta(z, \mathbb{C}^n \setminus \Omega)$, $z \in \hat{K}_\Omega$, which in particular implies

$$v(\tau) \le \operatorname{Re} p(\tau), \quad |\tau| \le \varepsilon.$$

This $\Rightarrow$ v is SH in $|\tau| < \varepsilon$ $\Rightarrow$ $u \in PSH(\Omega)$. $\square$

Def: Let $\Omega \subseteq \mathbb{C}^n$ and $K \subset\subset \Omega$. The <u>PSH(Ω)-hull</u>, $\hat{K}^P_\Omega \subseteq \Omega$ is

$$\hat{K}^P_\Omega = \{z \in \Omega : u(z) \le \sup u(\zeta),\ \forall u \in PSH(\Omega)\}.$$

Def: Let $\Omega \subseteq \mathbb{C}$ and $K \subset\subset \Omega$. The ...

$$\hat{K}^P_\Omega = \{ z \in \Omega : u(z) \le \sup_{\zeta \in K} u(\zeta),\ \forall u \in PSH(\Omega) \}.$$

Rem. Note that $f \in \mathcal{O}(\Omega) \Rightarrow u = \log|f| \in PSH(\Omega)$. Thus, if $z \in \hat{K}^P_\Omega$, then $\log|f(z)| \le \sup_K \log|f| \Rightarrow |f(z)| \le \sup_K |f| \Rightarrow z \in \hat{K}_\Omega$.

$\Rightarrow \boxed{\hat{K}^P_\Omega \subseteq \hat{K}_\Omega}$.

W/ $\delta(z, \mathbb{C}^n \setminus \Omega)$ as above (cont. + $\delta(tz) = |t|\delta(z)$, $t \in \mathbb{C}$):

Thm 6. Let $\Omega \subseteq \mathbb{C}^n$. TFAE:

(i) $u(z) = -\log \delta(z, \mathbb{C}^n \setminus \Omega)$ is $PSH(\Omega)$ or $\Omega = \mathbb{C}^n$.

(ii) $\exists$ cont. $PSH(\Omega)$ fcn $v(z)$ s.t. $\overline{\Omega}_c := \{ z \in \Omega : v(z) \le c \} \subset\subset \Omega$, $\forall c \in \mathbb{R}$.

(iii) $\forall K \subset\subset \Omega$, $\hat{K}^P_\Omega \subset\subset \Omega$.

Rem. A consequence is that (i) holds for all δ if it holds for some δ.

Pf. (i) $\Rightarrow$ (ii). Take $v(z) = |z|^2 - \log \delta(z, \mathbb{C}^n \setminus \Omega)$ if $\Omega \ne \mathbb{C}^n$ and $v(z) = |z|^2$ if $\Omega = \mathbb{C}^n$.

(ii) $\Rightarrow$ (iii). Let $K \subset\subset \Omega$ and $c = \sup_K v$. If $z \in \hat{K}^P_\Omega \Rightarrow v(z) \le \sup_K v = c \Rightarrow z \in \overline{\Omega}_c \Rightarrow$
$\hat{K}^P_\Omega \subseteq \overline{\Omega}_c \subset\subset \Omega$. $\hat{K}^P_\Omega$ closed in Ω (by u.s.c., $\lim_{z \to z^0} u(z) \le u(z^0)$) $\Rightarrow$
$\hat{K}^P_\Omega \subset\subset \Omega$, i.e. (iii).

For (iii) $\Rightarrow$ (i), we need the following lemma:

Lemma 1. If $f : \Omega \to \Omega'$ is holom. map and $u \in PSH(\Omega')$, then $u \circ f \in PSH(\Omega)$.

Pf. If $u \in C^2$, then this follows from chain rule, since $v = u \circ f$ satisfies

$$\sum_{i,j} \frac{\partial^2 v}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j = \sum_{i,j} \left(\sum_{k,l} \frac{\partial^2 u}{\partial \zeta_k \partial \bar{\zeta}_l} f_{k,z_i} \overline{f_{l,z_j}} \right) w_i \bar{w}_j = \sum_{k,l} \frac{\partial^2 u}{\partial \zeta_k \partial \bar{\zeta}_l} \sum_{i,j} \left(f_{k,z_i} w_i \right) \left(\overline{f_{l,z_j} w_j} \right)$$

$$= \sum_{k,l} \frac{\partial^2 u}{\partial \zeta_k \partial \bar{\zeta}_l} w'_k \overline{w'_l}$$ w/ $w'_k = \sum_i f_{k,z_i} w_i$. Thus, $\sum_{i,j} \frac{\partial^2 v}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j \ge 0$.

If $u \in PSH$ but not C^2, let $u_\varepsilon \in C^2 \cap PSH$ be regularization, $\varepsilon > 0$. Since $u_\varepsilon \searrow u$, $v_\varepsilon = u_\varepsilon \circ f \searrow v$. By above $v_\varepsilon \in PSH$. Now, conclusion follows from Ex. If $v_\varepsilon \searrow v$, $v_\varepsilon \in PSH$, $\Rightarrow v \in PSH$. ∎

(iii) $\Rightarrow$ (i). Pick $z^0 \in \Omega$, $w \in \mathbb{C}^n$ and let $\tau \in \mathbb{C}$, $|\tau| \le \varepsilon$, s.t.

conclusion follows from Ex. + $\forall \varepsilon$ ...

(iii) ⇒ (i). Pick $z^0 \in \Omega$, $w \in \mathbb{C}^n$, and let $\tau \in \mathbb{C}$, $|\tau| \leq \varepsilon$, s.t.

$\overline{D} = \{z^0 + w\tau : |\tau| \leq \varepsilon\} \subset\subset \Omega$. Pick holom. poly. $f(\tau)$,

$f(\tau) = Q(z^0 + \tau w)$ some holom. poly $Q(z)$, s.t.

$$-\log \delta(z^0 + \tau w, \mathbb{C}^n \setminus \Omega) \leq \operatorname{Re} f(\tau), \quad |\tau| = \varepsilon. \iff$$

$$\delta(z^0 + \tau w, \mathbb{C}^n \setminus \Omega) \geq |e^{-f(\tau)}|, \quad |\tau| = \varepsilon \qquad (1)$$

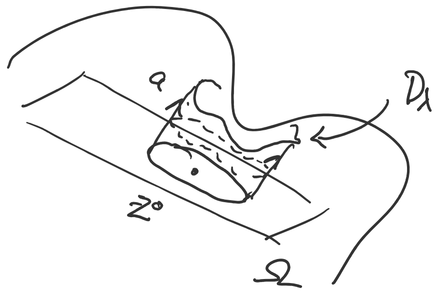
Pick $a \in \mathbb{C}^n$, $\delta(a) < 1$, and consider $F_\lambda(\tau) = z^0 + \tau w + \lambda a e^{-f(\tau)}$, $|\tau| \leq \varepsilon$,

and $0 \leq \lambda \leq 1$. Let $\overline{D_\lambda} = \{F_\lambda(\tau) : |\tau| \leq \varepsilon\}$. Have $\overline{D_0} = \overline{D} \subset\subset \Omega$.

Let $\Lambda = \{0 \leq \lambda \leq 1 : \overline{D_\lambda} \subset\subset \Omega\}$. We have $0 \in \Lambda$. Clearly, Λ is open. (by uniform cont.)

Claim: Λ is closed.

Consider $K = \{z^0 + \tau w + a\lambda e^{-f(\tau)} : |\tau| = \varepsilon, 0 \leq \lambda \leq 1\}$. By (1), $K \subset\subset \Omega$.



Let $u \in PSH$, $\lambda \in \Lambda$, then $v(\tau) = u(z^0 + \tau w + a\lambda e^{-f(\tau)})$ is SH on D_λ by Lemma 1.

⇒ for $|\tau_0| < \varepsilon$, by MMP, $v(\tau_0) \leq \sup_{|\tau| = \varepsilon} v(\tau) \leq \sup_K u$ ⇒

$z^0 + \tau_0 w + a\lambda e^{-f(\tau_0)} \in \hat{K}^P_\Omega \subset\subset \Omega \Rightarrow \overline{D_\lambda} \subset\subset \hat{K}^P_\Omega \subset\subset \Omega$, $\forall \lambda \in \Lambda$.

By continuity, if $\lambda_n \in \Lambda$, $\lambda_n \to \lambda$, $\Rightarrow \overline{D_\lambda} \subset\subset \hat{K}^P_\Omega \Rightarrow \lambda \in \Lambda$. ⇒ claim.

Thus, $\Lambda \neq \emptyset$, Λ open + closed in $[0,1] \Rightarrow \Lambda = [0,1] \Rightarrow \overline{D_1} \subset\subset \Omega$, i.e. for any $|\tau| \leq \varepsilon$, $z^0 + \tau w + a e^{-f(\tau)} \in \Omega$. Moreover, since $a \in \mathbb{C}^n$ is arbitrary as long as $\delta(a) < 1$, we may conclude that the "δ-ball" $\{z^0 + \tau w + a e^{-f(\tau)} : a \in \mathbb{C}^n, \delta(a) < 1\}$ centered at $z^0 + \tau w$ is contained in Ω; i.e.

$$\delta(z^0 + \tau w, \mathbb{C}^n \setminus \Omega) \geq |e^{-f(\tau)}|, \quad \forall |\tau| \leq \varepsilon.$$

centered at $z^0+\tau w$ is contained in Ω; i.e.

$$\delta(z^0+\tau w, \mathbb{C}^n\setminus\Omega) \geq |e^{-f(\tau)}|, \ \forall |\tau|\leq\varepsilon.$$

$\Longleftrightarrow \quad -\log\delta(z^0+\tau w, \mathbb{C}^n\setminus\Omega) \leq \operatorname{Re} f(\tau), \ |\tau|\leq\varepsilon \Rightarrow -\log\delta(z, \mathbb{C}^n\setminus\Omega) \in PSH(\Omega).$

$\boxtimes$

Def. $\Omega\subseteq\mathbb{C}^n$ is $\overbrace{\underline{\text{pseudoconvex}}}^{(\psi\text{cvx})}$ if either (thus, all) conditions (i)-(iii) in Thm 6 holds.

Rem. The proof of (iii) $\Rightarrow$ (i) in Thm 6 is an example of the

Continuity Principle. Let $\Omega\subseteq\mathbb{C}^n$ be ψcvx. Let $f_\lambda : \overline{\mathbb{D}}\subseteq\mathbb{C}\to\mathbb{C}^n$ a cont. family $\{f_\lambda\}_{\lambda\in[0,1]}$ of holom. disks, $\left(f_\lambda\in\mathcal{O}(\mathbb{D})\cap\mathcal{C}(\overline{\mathbb{D}}),\ f(z,\lambda)\in\mathcal{C}(\overline{\mathbb{D}}\times[0,1])\right)$ s.t.

(1) $f_0(\overline{\mathbb{D}})\subset\subset\Omega$

(2) $f_\lambda(\partial\mathbb{D})\subset\subset\Omega, \ \lambda\in[0,1]$

Then, $\{f_\lambda(\overline{\mathbb{D}}) : \lambda\in[0,1]\}\subset\subset\Omega$.

Pf: Essentially done in (iii) $\Rightarrow$ (i) above. $\boxtimes$